A GROUP ACTION ON LOSEV-MANIN COHOMOLOGICAL FIELD THEORIES

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Abstract. We discuss an analog of the Givental group action for the space of solutions of the commutativity equation. There are equivalent formulations in terms of cohomology classes on the Losev-Manin compactifications of genus 0 moduli spaces; in terms of linear algebra in the space of Laurent series; in terms of differential operators acting on Gromov-Witten potentials; and in terms of multi-component KP tau-functions. The last approach is equivalent to the Losev-Polyubin classification that was obtained via dressing transformations technique.

Contents

1. Introduction

Frobenius manifolds are among the most important notions in modern mathematics and mathematical physics, capturing the universal structure hidden behind different notions in enumerative geometry, singularity theory, integrable hierarchies, and string theory [7, 8, 13, 25]. Roughly speaking, a Frobenius structure on a manifold is an associative product in every fiber

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of the tangent bundle, subject to some integrability and homogeneity conditions. A precise definition involves the celebrated WDVV equation [27, 6] that reflects the topology of the Deligne-Mumford compactification of moduli space of genus 0 curves and makes the whole theory of Frobenius manifolds so interesting and beautiful.

There are many different methods developed in the course of study of Frobenius manifolds. One of the most promising ones is due to Givental, who constructed a group action on the space of Frobenius manifolds [11, 12]. It allows, roughly speaking, to transfer known results from some particularly simple Frobenius manifolds to the other ones that are in the same orbit of the Givental group action. It was used in many different applications; some references are [1, 4, 5, 9, 18, 26].

Another method that was proposed by Losev [20, 21] is based on the idea that a part of the structure of a Frobenius manifold can be reconstructed, under certain assumptions, from a simpler structure: namely, a germ of a pencil of flat connections. It is used in many works, some recent examples being [2, 7, 23, 24]. A precise definition involves the so-called commutativity equation that reflects the topology of a different compactification of the moduli space of genus 0 curves [22]. This is a sort of linearization of the notion of Frobenius manifolds, and at the level of the underlying solutions of the commutativity equation many concepts and theorems about Frobenius manifolds appear to be much simpler.

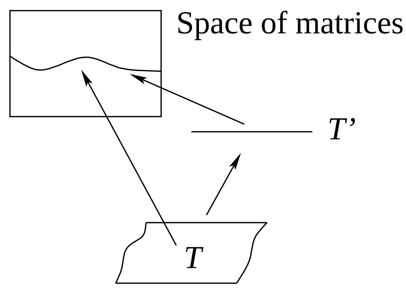
In this paper we discuss an analog of Givental's group action on the space of solutions of the commutativity equation. We describe it from the point of view of cohomology classes on the Losev-Manin moduli spaces, in terms of differential operators on formal matrix Gromov-Witten potential, and in terms of a linear algebraic interpretation of the descendant version of commutativity equation. We also link it to the Losev-Polyubin classification of solutions of the commutativity equation in terms of τ-functions of multicomponent KP hierarchies [14, 15, 19].

We hope that our results also help with further understanding of the Givental group action on the space of Frobenius manifolds and shed light on some of the ideas behind Givental's theory.

1.1. **The commutativity equation.** Let $M(t)$ be a complex analytic matrix-valued function in several complex variables $t = (t^1, \dots, t^N)$. The matrices are of size $m \times m$. The commutativity equation on this function reads $dM \wedge dM = 0$. The function $M(t)$ satisfies the commutativity equation if and only if the matrices $\partial M/\partial t^i$ and $\partial M/\partial t^j$ commute at every point t for every i and j. In this paper we study the solutions of this equation, more precisely, germs of solutions at the origin $t = 0$.

Definition 1.1. A germ of solution of the commutativity equation is called *nonsingular* if the map $M(t)$ is a composition of a submersion with an immersion (see the figure below).

In this paper we restrict ourselves to nonsingular solutions.



Note that although the space of matrices has dimension m^2, the image of T in it is of dimension at most m. Indeed, the tangent space to this image at any given point is composed of mutually commuting matrices.

Note further that it makes sense to study the solution of the commutativity equation directly on T'. Indeed, going from T' to T means just adding several coordinates to the parameter space on which the matrix M does not depend. Therefore we will usually assume that M is an immersion.

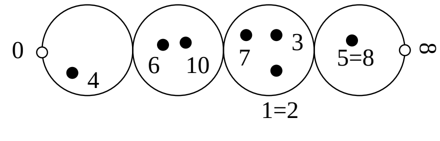
1.2. **Pencils of flat connections.** Solutions of the commutativity equation can be described in more intrinsic terms. First, the coordinates $t^1, \dots, t^N$ must be viewed as local coordinates on a base complex manifold T. Indeed, the commutativity equation is preserved by any biholomorphic change of variables t. Over T we have a trivial vector bundle of rank m with the trivial flat connection d. If M is a solution of the commutativity equation, then this vector bundle possesses a whole pencil of flat connections depending on a parameter z. They are given by

$$\nabla_z = d - \frac{1}{z} dM.$$

1.3. **The Losev-Manin moduli spaces.** In [22] A. Losev and Yu. Manin introduced a new compactification of $\mathcal{M}_{0,n+2}$ denoted by L_n. The marked points do not play a symmetric role in this compactification: two "white" marked points, labeled 0 and ∞, are not allowed to coincide with each other or with any other marked points; the remaining $n \geq 1$ "black" marked points can coincide with each other.

Definition 1.2. A *Losev-Manin stable curve* is a nodal curve that has the form of a chain of spheres composed of one or more spheres; the leftmost sphere of the chain contains a white marked point labeled 0, the rightmost sphere of the chain contains a white marked point labeled ∞; every sphere contains at least one black marked point; white points and nodes do not coincide with each other or with black marked points, but black marked points are allowed to coincide.

The *Losev-Manin space* L_n is the moduli space of Losev-Manin stable curves with n numbered black points.



The points of a boundary divisor of L_n correspond to curves with at least one node dividing the set of black points into two parts. Thus the boundary

divisors of L_n correspond to ordered partitions of the set of black points into two non-empty subsets. Every boundary divisor is isomorphic to $L_p \times L_q$ with $p+q=n$.

1.4. **The Losev-Manin cohomological field theories.** Recall that an ordinary *cohomological field theory* (CohFT) on a vector space V is given by a nondegenerate bilinear symmetric form η on V and a collection of maps $\omega_n : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{0,n}, \mathbb{C})$ satisfying certain properties.

Now let V and T be two complex vector spaces. Intuitively, V is associated with the white marked points, while T is associated with the black ones.

Definition 1.3. A *Losev-Manin cohomological field theory* is a system of maps

$$\alpha_n : T^{\otimes n} \to H^*(L_n, \mathbb{C}) \otimes \mathrm{End}(V)$$

satisfying the following properties. (i) The maps are S_n-equivariant with respect to the renumbering of the marked points and a simultaneous permutation of the factors in $T^{\otimes n}$. (ii) The restriction of α_n to a boundary divisor $L_p \times L_q \subset L_n$ is the composition of α_p and α_q.

Note that the space $\mathrm{End}(V)$ being self-dual, we could have moved the tensor factor $\mathrm{End}(V)$ to the left-hand side of the map α. But our convention is often easier to work with.

Losev-Manin cohomological field theories arise, in particular, as an example of extension of the Gromov-Witten invariants of Kähler manifolds. This construction was developed in [3] in the much more general setting of moduli spaces of curves and maps with weighted stability conditions.

Let $\omega_n : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{0,n}, \mathbb{C})$ be a CohFT in the usual sense [16]. Define $\beta_n : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{0,n}, \mathbb{C}) \otimes \mathrm{End}(V)$ by moving the last two factors V of ω_{n+2} (corresponding to the marked point $n+1$ and $n+2$) to the right-hand side of the map and dualizing the last factor with the bilinear form η. Let $p_n : \overline{\mathcal{M}}_{0,n+2} \to L_n$ be the natural morphisms.

Proposition 1.4. *$\alpha_n = (p_n)_*(\beta_n)$ is a Losev-Manin CohFT with $T = V$.*

Proof. The S_n-equivariance of α_n follows from the S_n-equivariance of β_n, which follows from the S_{n+2}-equivariance of ω_{n+2}.

The preimage $p_n^{-1}(L_p \times L_q)$ of a boundary divisor equals $\overline{\mathcal{M}}_{0,p+2} \times \overline{\mathcal{M}}_{0,q+2}$. Therefore, by the projection formula,

$$\alpha_n|_{L_p \times L_q} = ((p_n)_*(\beta_n))|_{L_p \times L_q} = (p_n)_*(\beta_n|_{\overline{\mathcal{M}}_{0,p+2} \times \overline{\mathcal{M}}_{0,q+2}})$$

$$= (p_n)_*(\beta_p \circ \beta_q)) = \alpha_p \circ \alpha_q.$$

□

Note that the other way round there is no simple way to construct a usual CohFT starting from a Losev-Manin CohFT.

1.5. **Gromov-Witten potentials.** To a Losev-Manin CohFT we can assign matrix Gromov-Witten potentials in the following way.

Let (α_n) be a Losev-Manin CohFT with underlying vector spaces V and T.

Definition 1.5. We call *matrix Gromov-Witten potentials* the endomorphisms $M_{a,b}(t) \in \mathrm{End}(V)$ given by

$$M_{a,b}(t) = \sum_{n \geq 1} \frac{1}{n!} \int\limits_{L_n} \alpha_n(t \otimes \cdots \otimes t)\ \psi_0^a \psi_\infty^b$$

for $a, b = 0, 1, \ldots$.

$M_{a,b}$ is a formal power series in variables t^i, the degree n part corresponding to the contribution of L_n. Denote by $\dot{M}_{a,b}$ the $\mathrm{End}(V)$-valued differential form $d_t M_{a,b}$ on T.

Proposition 1.6. *The matrix potentials $M_{a,b}$ satisfy the following master equations:*

$$\begin{aligned} \dot{M}_{a+1,b} &= M_{a,0}\, \dot{M}_{0,b}, \\ \dot{M}_{a,b+1} &= \dot{M}_{a,0}\, M_{0,b}, \\ M_{a+1,b} + M_{a,b+1} &= M_{a,0}\, M_{0,b}. \end{aligned}$$

Proof. The first two equations follow from the expressions of ψ_0 and ψ_∞ as sums of boundary divisors. The last equation follows from the equality $\psi_0 + \psi_\infty = \delta$, where δ is the cohomology class Poincaré dual to the boundary of L_n. □

Definition 1.7. The family of matrix Gromov-Witten potentials $(M_{a,b})_{a,b \geq 0}$ is called a *tower*.

The matrix Gromov-Witten potentials can be regrouped into a unique power series depending on variables $q_0, q_1, \dots \in V$ and $p_0, p_1, \dots \in V^*$.

Definition 1.8. The *full Gromov-Witten potential* associated to a Losev-Manin CohFT is the power series

$$F(p, q, t) = \sum_{a,b \geq 0} M_{a,b}(t) p_a q_b,$$

where $q = (q_0, q_1, \ldots)$ and $p = (p_0, p_1, \ldots)$.

Let $(M_{a,b})$ be a tower of matrix Gromov-Witten potentials associated with a Losev-Manin CohFT.

Proposition 1.9. *$M_{0,0}$ is a solution of the commutativity equations.*

Proof. One of the master equations reads $dM_{1,0} = M_{0,0}\, dM_{0,0}$. Taking a differential (with respect to t) we obtain $0 = dM_{0,0} \wedge dM_{0,0}$. □

Consider the trivial vector bundle $V \times T \to T$ with a pencil of flat connections $\nabla_z = d - \dot{M}_{0,0}/z$.

Proposition 1.10.

$$J(z) = I + \sum_{b=0}^{\infty} M_{0,b}\, z^{-(b+1)},$$

where I is the identity matrix, is a basis of flat sections of the connections ∇_z.

Proof. $\nabla_z J(z) = (\dot{M}_{0,0} - \dot{M}_{0,0})\, z^{-1} + \sum_{b\geq 0} (\dot{M}_{0,b+1} - \dot{M}_{0,0} M_{0,b})\, z^{-(b+2)} \overset{\mathrm{ME}}{=} 0$, where the equality ME follows from the first two master equations. $\square$

Example 1.11. If $\dim V = 1$, the matrix $M_{0,0}$ automatically commutes with its differential $\dot{M}_{0,0}$. Therefore the equation

$$\nabla_z J = 0 \Longleftrightarrow \dot{J} = \frac{1}{z} \dot{M}_{0,0} J$$

has an explicit solution: $J = e^{M_{0,0}/z}$. Thus $M_{0,b} = M_{0,0}^{b+1}/(b+1)!$. It follows that

$$M_{a,b} = \frac{M_{0,0}^{a+b+1}}{a!\, b!\, (a+b+1)}.$$

Indeed, the third master equation reads $M_{a,b} = M_{a-1,0} M_{0,b} - M_{a-1,b+1}$. Assuming by induction that the formula for $M_{a,0}$, $M_{0,b}$, and $M_{a-1,b+1}$ is valid, we get

$$M_{a,b} = \frac{M_{0,0}^{a}}{a!} \cdot \frac{M_{0,0}^{b+1}}{(b+1)!} - \frac{M_{0,0}^{a+b+1}}{(a-1)!\, (b+1)!\, (a+b+1)} = \frac{M_{0,0}^{a+b+1}}{a!\, b!\, (a+b+1)}.$$

$\square$

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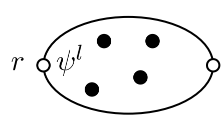
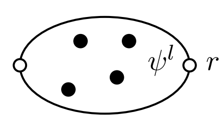
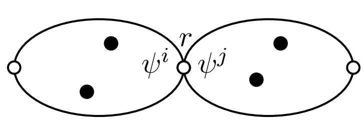
2. The upper triangular group

Consider a Losev-Manin CohFT (α_n). Before introducing the group action, let us ask the following (presently unmotivated) question: given an endomorphism r of V, what are the natural ways to increase the degree of each α_n by an integer l using r exactly once? The answer is provided in the following picture that represents all natural ways to do that.

(A) 

(B) 

(C) 

$(i + j = l - 1)$

These pictures represent the following composition maps:

$$\text{(A)} \qquad T^{\otimes n} \xrightarrow{\alpha_n} H^*(L_n) \otimes \mathrm{End}(V) \xrightarrow{\psi_0^l \otimes (r\,\circ)} H^*(L_n) \otimes \mathrm{End}(V),$$

$$\text{(B)} \qquad T^{\otimes n} \xrightarrow{\alpha_n} H^*(L_n) \otimes \mathrm{End}(V) \xrightarrow{\psi_\infty^l \otimes (\circ\, r)} H^*(L_n) \otimes \mathrm{End}(V),$$

$$\text{(C)} \qquad T^{\otimes n} \simeq T^{\otimes p} \otimes T^{\otimes q} \xrightarrow{\alpha_p \otimes \alpha_q} H^*(L_p) \otimes H^*(L_q) \otimes \mathrm{End}(V) \otimes \mathrm{End}(V)$$
$$\xrightarrow{\left(\mathrm{Gysin} \circ (\psi')^{\mathrm{i}} (\psi'')^{\mathrm{j}}\right) \otimes (\circ\, r\, \circ)} H^*(L_n) \otimes \mathrm{End}(V).$$

Let us denote these composition maps by $A_l(r)$, $B_l(r)$, and $C_l^{(i,j|I,J)}(r)$, where $I \sqcup J = \{1, \ldots, n\}$.

Now we can describe first a Lie algebra action and then a Lie group action on Losev-Manin cohomological field theories.

Consider the Lie group G_+ of formal power series $R(z)$ with values in $\mathrm{End}(V)$ such that $R(0) = \mathrm{id}$. Its Lie algebra $\mathcal{G}_+$ is composed of formal power series $r(z)$ with coefficients in $\mathrm{End}(V)$ such that $r(0) = 0$.

Let $r = \sum_{l \geq 1} r_l z^l$ be an element of $\mathcal{G}_+$.

Definition 2.1. Define the action of r on a Losev-Manin CohFT by the formula

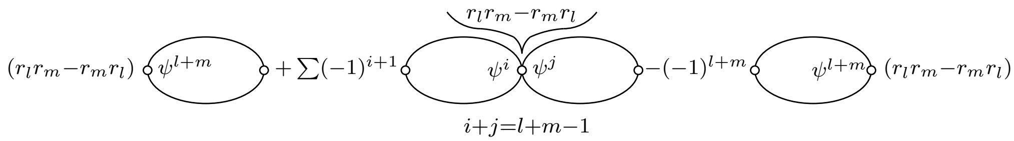
$$(r.\alpha)_n = \sum_{l \geq 1} \left[A_l(r_l) \; - \; (-1)^l B_l(r_l) \; + \sum_{\substack{i+j=l-1 \\ I \sqcup J = \{1,\ldots,n\},\, |I|,|J| \geq 1}} (-1)^{i+1} C_l^{(i,j|I,J)}(r_l) \right]$$

Theorem 2.2. *The action of $\mathcal{G}_+$ is a well-defined Lie algebra action. It lifts to a group action of G_+ that takes every Losev-Manin CohFT to a Losev-Manin CohFT.*

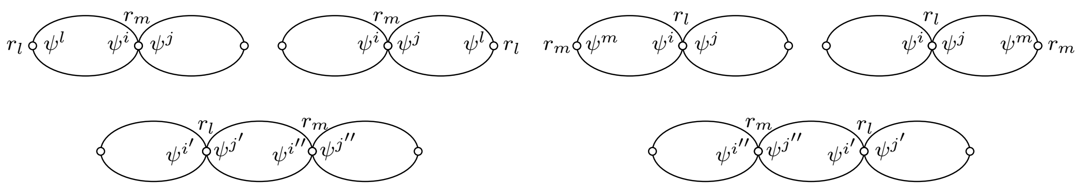
Proof. First of all, let us check that we can exponentiate the action of $r \in \mathcal{G}_+$. Indeed, as we have already remarked, the action of r_l adds $l \geq 1$ to the degree of its ingredients. Thus $(r^k.\alpha)_n$ vanishes for $k > \dim L_n = n - 1$. We conclude that $e^r.\alpha$ is well-defined, since each of its components is the sum of a finite number of terms.

Therefore the action of $R \in G_+$ can be defined as the exponential of the action of $r = \ln R$.

Now we check that the action of $\mathcal{G}_+$ is compatible with the Lie algebra structure. First of all, note that the action of r on a Losev-Manin CohFT is not linear. Indeed, the term $C^{(i,j|I,J)}$ involves a product of α_p and α_q. Therefore, as we compute the commutator of two actions, we will have to apply the first action to α_p (without acting of α_q) then to α_q (without acting on α_p), then add up the results and compose with the second action. We have $[r_l z^l, r_m z^m] = (r_l r_m - r_m r_l) z^{m+l}$. The action of the right-hand side of this equality is represented in the following picture (with the same conventions as above):

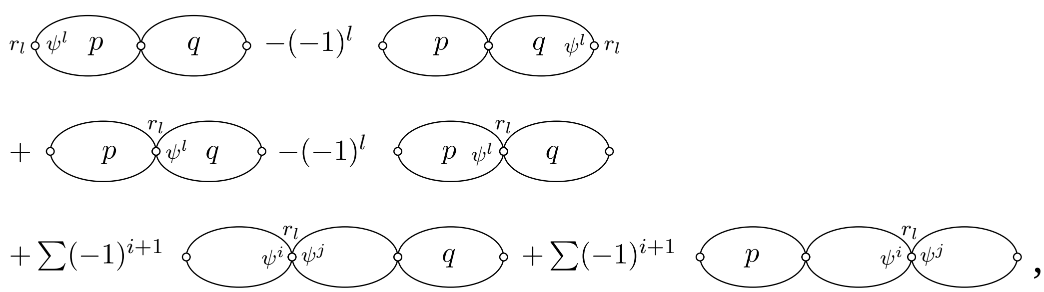


It is also easy to see that the action of the left-hand side is given by the same formula, after the cancellation of the terms of the form:



To understand how the middle term in the previous formula appears when we compute the commutator of two actions it is useful to remark that for $i+j=m+l-1$ we have either $i \geq l$ or $j \geq m$, but not both. And similarly either $i \geq m$ or $j \geq l$, but not both. This explains why every pair (i,j) appears exactly once with coefficient $r_l r_m$ and once with coefficient $r_m r_l$.

Finally, we must check that the action of G_+ takes a Losev-Manin CohFT to a Losev-Manin CohFT. In other words, we need to check that the restriction of $(r.\alpha)_n$ to a boundary divisor $L_p \times L_q$ equals $\alpha_p \times (r.\alpha)_q + (r.\alpha)_p \times \alpha_q$. A simple computation shows that both are actually equal to



where the summation over l is assumed.

An explanation is in order as to how the third and the fourth terms appear in the restriction of $(r.\alpha)_n$ to $L_p \times L_q$. These terms arise when the partition $I \sqcup J$ of the n marked points in $C^{(i,j|I,J)}$ is exactly the same as in the boundary divisor $L_p \times L_q$. The self-intersection of this boundary divisor equals $-(L_p \times L_q)(\psi' + \psi'')$. Multiplied by $\sum(-1)^{i+1}(\psi')^i(\psi'')^j$ this gives $(-1)^{l+1}(\psi')^l + (\psi'')^l$ as shown in the figure. The other terms are straightforward. $\square$

Proposition 2.3. *The action of r on the matrix Gromov-Witten potentials is given by*

$$(1)\quad (r.M)_{a,b} = \sum_{l \geq 1} \left[r_l M_{a+l,b} - (-1)^l M_{a,b+l} r_l + \sum_{i+j=l-1} (-1)^{i+1} M_{a,i} r_l M_{j,b} \right].$$

To formulate the next proposition we choose a basis of V and a dual basis of V^*. The indices μ and ν run over these bases and the summation over repeated indices is assumed.

Proposition 2.4. *The action of r on the exponent of the full Gromov-Witten potential is given by the differential operator*

$$\widehat{r} = \sum_{l\geq 1}\Bigg[\sum_{a\geq 0}(r_l)^\nu_\mu p_{a,\nu}\frac{\partial}{\partial p_{a+l,\nu}} - (-1)^l\sum_{b\geq 0}(r_l)^\nu_\mu q^\mu_b\frac{\partial}{\partial q^\nu_{b+l}} + \sum_{i+j=l-1}(-1)^{i+1}(r_l)^\nu_\mu\frac{\partial^2}{\partial q^\nu_i\partial p_{j,\mu}}\Bigg]. \tag{2}$$

The claims of both propositions follow immediately from the definition of the action of r on a Losev-Manin CohFT.

Example 2.5. Consider the tower of matrix Gromov-Witten potentials from Example 1.11:

$$M_{a,b} = \frac{M_{0,0}^{a+b+1}}{a!\,b!\,(a+b+1)},$$

$\dim V = 1$. Every series r acts trivially on this tower. This follows from the combinatorial identity:

$$\frac{1}{(a+l)!\,b!\,(a+b+l+1)} - \frac{(-1)^l}{a!\,(b+l)!\,(a+b+l+1)} + \sum_{i+j=l-1}\frac{(-1)^{i+1}}{a!\,b!\,i!\,j!\,(a+i+1)\,(b+j+1)} = 0$$

for any $a,b\geq 0$, $l\geq 1$.

Proposition 2.6. *The action of G_+ preserves the spectrum of $\dot{M}_{0,0}$.*

Proof. Since $\dot{M}_{0,0}$ is a matrix of differential 1-forms on T, its spectrum is also a collection of N differential 1-forms.

First of all, note that Definition 2.1 and Propositions 2.3 and 2.4 define a valid action for $r = r_0 + r_1 z + \dots$ even if $r_0 \neq 0$. In particular,

$$(r_0.M)_{a,b} = r_0 M_{a,b} - M_{a,b} r_0.$$

We claim that $d_t(r.M)_{0,0}$ is equal to the commutator $[((r/z).M)_{0,0}, d_t M_{0,0}]$. The assertion of the proposition follows immediately from this equality. The equality itself is obtained by the following computations:

$$d_t(r.M)_{0,0} =$$

$$\sum_{l\geq 1}\left(r_l\,\dot{M}_{l,0} - (-1)^l\,\dot{M}_{0,l}\,r_l + \sum_{i+j=l-1}(-1)^{i+1}(\dot{M}_{0,i}\,r_l\,M_{j,0} + M_{0,i}\,r_l\,\dot{M}_{j,0})\right) =$$

$$\sum_{l\geq 1}\left((-1)^{l+1}\,\dot{M}_{0,l}\,r_l + \sum_{i+j=l-1}(-1)^{i+1}\dot{M}_{0,i}\,r_l\,M_{j,0}\right) +$$

$$\sum_{l\geq 1}\left(r_l\,\dot{M}_{l,0}+\sum_{i+j=l-1}(-1)^{i+1}\,M_{0,i}\,r_l\,\dot{M}_{j,0}\right)\overset{\mathrm{ME}}{=}$$

$$\sum_{l\geq 1}\left(-\dot{M}_{0,0}r_lM_{0,l-1}+(-1)^{l-1}\dot{M}_{0,0}M_{0,l-1}r_l)+\sum_{\substack{i+j=l-1\\ i\geq 1}}(-1)^{i+1}\dot{M}_{0,0}M_{0,i-1}r_lM_{j,0}\right)+$$

$$\sum_{l\geq 1}\left(r_lM_{l-1,0}\dot{M}_{0,0}-(-1)^{l-1}M_{l-1,0}r_l\dot{M}_{0,0}+\sum_{\substack{i+j=l-1\\ j\geq 1}}(-1)^{i+1}\,M_{0,i}\,r_l\,M_{j-1,0}\,\dot{M}_{0,0}\right)=$$

$$=[((r/z).M)_{0,0},\dot{M}_{0,0}].$$

□

3. The lower triangular group

Now consider the Lie group G_- of formal power series $S(z^{-1})$ with values in $\mathrm{End}(V)$ such that $S=\mathrm{id}$ at $1/z=0$. Its Lie algebra $\mathcal{G}_-$ is composed of formal power series $s(z^{-1})$ with coefficients in $\mathrm{End}(V)$ such that $s=0$ at $1/z=0$. This group does not act on Losev-Manin cohomological field theories, but only on Gromov-Witten potentials.

Let $s=\sum_{l\geq 1}s_lz^{-l}$ be an element of $\mathcal{G}_-$.

Definition 3.1. The action of s on the matrix Gromov-Witten potentials is given by

$$(s.M)_{a,b}=\sum_{l\geq 1}\left[s_lM_{a-l,b}\ -\ (-1)^lM_{a,b-l}s_l\ +(-1)^b\,\delta_{a+b+1,l}\,s_l\right],$$

where by convention a matrix Gromov-Witten potential vanishes if one of its indices is negative.

The action of s on the exponent e^F of the full Gromov-Witten potential is given by the differential operator

$$\widehat{s}=\sum_{l\geq 1}\left[\sum_{a\geq 0}(s_l)^\nu_\mu p_{a+l,\nu}\frac{\partial}{\partial p_{a,\mu}}-(-1)^l\sum_{b\geq 0}(s_l)^\nu_\mu q^\mu_{b+l}\frac{\partial}{\partial q^\nu_b}\right.$$
$$\left.+\sum_{i+j=l-1}(-1)^j\,(s_l)^\nu_\mu\,p_{i,\nu}\,q^\mu_j\right].\tag{3}$$

It is obvious that both definitions are equivalent.

Theorem 3.2. *Definition 3.1 gives a well-defined Lie algebra action of $\mathcal{G}_-$ on Gromov-Witten potentials. It preserves the master equations and can be integrated to a well-defined group action of G_-.*

Proof. The action of s_l decreases the sum of indices $a+b$ of a matrix Gromov-Witten potential by l. Therefore only a finite number of actions of s can be applied in succession before their contributions to $M_{a,b}$ become identically vanishing. We conclude that the action of e^s is well-defined, since we only need a finite number of steps to compute $(e^s.M)_{a,b}$ for any given a,b.

Let us check that the action is compatible with the Lie bracket. Computing the commutator of the operators $\widehat{s_l z^l}$ and $\widehat{s_m z^m}$ we obtain

$$[\widehat{s_l z^l}, \widehat{s_m z^m}] =$$
$$\sum_{a\geq 0}[s_l,s_m]^\nu_\mu\, p_{a+l+m,\nu}\frac{\partial}{\partial p_{a,\nu}} - (-1)^{l+m}\sum_{b\geq 0}[s_l,s_m]^\nu_\mu\, q^\mu_{b+l+m}\frac{\partial}{\partial q^\nu_b}$$
$$+\sum_{i+j=l+m-1}(-1)^j\,[s_l,s_m]^\nu_\mu\, p_{i,\nu}\, q^\mu_j,$$

which is indeed the action of $[s_l,s_m]z^{l+m}$.

Now let us check, for instance, that the action of G_- preserves the second master equation. We have

$$(s.\dot{M})_{a,b+1} = \sum_{l\geq 1}\Big[s_l\dot{M}_{a-l,b+1} - (-1)^l\dot{M}_{a,b+1-l}s_l\Big] =$$
$$\sum_{l\geq 1}\Big[s_l\dot{M}_{a-l,0}M_{0,b} - (-1)^l\dot{M}_{a,0}(M_{0,b-l}+\delta_{b+1,l})s_l\Big] =$$
$$(s.\dot{M})_{a,0}M_{0,b} + \dot{M}_{a,0}(s.M)_{0,b}.$$

We leave the analogous computations for the two other master equations to the reader. We encourage the reader to compute the commutator of two operators $\widehat{r_l z^l}$ and $\widehat{r_m z^m}$ from Proposition 2.4. □

Proposition 3.3. *Let $J(z) = I + \sum M_{0,b}z^{-(b+1)}$. The action of G_- preserves $\dot{M}_{0,0}$. The action of $s\in\mathcal{G}_-$ and of $S\in G_-$ on J are given by*

$$J.s = -J(z)\,s(-z),$$
$$J.S = J(z)\,S^{-1}(-z).$$

Proof. This follows immediately from the definition of the action. □

Corollary 3.4. *There is a unique $S\in G_-$ such that $(S.M)_{a,b}(0)=0$ for all a,b.*

Proof. Take $S(z) = J(-z)$. □

4. Group action orbits

We can sum up the results obtained so far as follows. A Losev-Manin CohFT determines a pencil of flat connections and a choice of a flat basis for every connection of the pencil. The lower half-group G_- acts on the choices of the flat basis, but preserves the connections themselves. The action of the upper half-group G_+ changes both the connections and the flat basis in a compatible way. In addition to these two groups, the group $\mathrm{Bihol}(T,0)$ of local biholomorphisms of the base T acts by coordinate changes.

Theorem 4.1. *Let $(M_{a,b})$ be a tower of matrix potentials. Assume that $d_tM_{0,0}$ is diagonalizable at the origin and its eigenvalues $\alpha_1,\dots,\alpha_N$ – linear forms in the variables t – are pairwise distinct. Then by a successive application of an element of the lower triangular group S and an element of the upper triangular group R one can arrive at a tower of pairwise commuting matrix potentials $(R.S.M)_{a,b}$.*

Proof. We choose the element S in such a way that $(S.M)_{a,b}(0) = 0$ for all a, b (see Corollary 3.4). From now on we will assume that the condition $M_{a,b}(0) = 0$ is satisfied from the start and we are looking for an upper triangular group element R such that the matrices $(R.M)_{a,b}$ commute.

Now we are going to prove the following property by induction on l: there exists a sequence of matrices $r_1, \dots, r_l \in \mathrm{End}(V)$ such that

$$(\exp(z^l r_l) \dots \exp(z^l r_1).M)_{0,0} = \text{diagonal} + O(t^{l+2}).$$

This property holds for $l = 0$, since, by out assumptions, $M_{0,0}(0) = 0$ and $d_t M_{0,0}(0)$ is diagonal.

The next two lemmas prepare the step of induction.

Lemma 4.2. *Let $(M_{a,b})$ be a tower of matrix potentials satisfying the master equations of Proposition 1.6, the condition $M_{a,b}(0) = 0$ for all a, b, and the condition*

$$M_{0,0} = \text{diagonal} + O(t^{l+1}).$$

Then

$$M_{a,b} = O(t^{a+b+1}) \qquad \text{and} \qquad M_{a,b} = \frac{M_{0,0}^{a+b+1}}{a!\, b!\, (a+b+1)} + O(t^{a+b+l+1}).$$

Proof. This is proved by induction on $a + b$ by integrating the master equations. □

Lemma 4.3. *Under the assumptions of Lemma 4.2 the diagonal matrix elements of $(z^l r_l.M)_{0,0}$ are $O(t^{l+2})$, while the off-diagonal matrix elements are given by*

$$\frac{(\alpha_\mu - \alpha_\nu)^{l+1} (r_l)_{\mu,\nu}}{(l+1)!} + O(t^{l+2}).$$

Proof. Just substitute the expression for $M_{a,b}$ from Lemma 4.2 into the formula that describes the action of r_l on $M_{0,0}$ (Proposition 2.3). □

Step of induction. Assume that $M_{0,0} = \text{diagonal} + O(t^{l+1})$. Let us study the term of order $l + 1$ in the Taylor expansion of $M_{0,0}$, that is, the first not necessarily diagonal term. Denote this term by X and its matrix elements by $X_{\mu,\nu}$.

Extract the degree l part in the equality $d_t M_{0,0} \wedge d_t M_{0,0} = 0$. We get $(\alpha_\mu - \alpha_\nu) \wedge dX_{\mu,\nu} = 0$ for all μ, ν. Since, by assumption, $\alpha_\mu - \alpha_\nu \neq 0$, this implies that $X_{\mu,\nu} = x_{\mu,\nu}(\alpha_\mu - \alpha_\nu)^l$ *for some constant* $x_{\mu,\nu}$.

Now we construct the matrix r_l by setting $(r_l)_{\mu,\nu} = -(l + 1)! x_{\mu,\nu}$ for $\mu \neq \nu$ and choosing the diagonal elements of r_l arbitrarily. According to Lemma 4.3, we have $(e^{r_l}.M)_{0,0} = \text{diagonal} + O(t^{l+2})$. Indeed, the action of r_l kills the off-diagonal elements of $M_{0,0}$ in degree $l + 1$, while the action of the higher powers of r_l only involves higher degree terms.

This prove the step of induction. It remains to note that the product $\cdots e^{z^l r_l} \cdots e^{z r_1}$ determines a well-defined element R of the upper triangular group, since every power of z only appears in a finite number of factors. The theorem is proved. □

Corollary 4.4. *Suppose that* $\dim T = \dim V$. *The joint action of the groups* G_-, G_+, $\mathrm{GL}(V)$, *and* $\mathrm{Bihol}(T, 0)$ *is transitive on the space of towers of*

matrix potentials such that $d_t M_{0,0}$ is diagonalizable at the origin and its eigenvalues span T^*.

Proof. First by an action of $\mathrm{GL}(V)$ we diagonalize $d_t M_{0,0}$ at the origin. Then by an action of $S \in G_-$ followed by an action of $R \in G_+$ we transform the tower of matrix potentials into a tower satisfying

$$M_{0,0}(0) = 0, \qquad M_{0,0} \text{ is diagonal}, \qquad M_{a,b} = \frac{M_{0,0}^{a+b+1}}{a!\, b!\, (a+b+1)}.$$

Finally, by a biholomorphic change of variables t we transform the matrix $M_{0,0}$ into its linear part (so that its matrix elements are linear forms in the variables t). $\square$

5. The commutativity equation and the loop space

In this section we give an interpretation of the commutativity equation in terms of linear algebra of the loop space of V (alternatively, it can be rewritten in terms of symplectic linear algebra of the loop space of $V + V^*$). This gives an alternative explanation as to why the loop group of $\mathrm{GL}(V)$ acts on the solutions of the commutativity equation.

5.1. **A special family of linear maps.** In this subsection we give an intermediate description in terms of linear algebra of the loop space of V.

Let $\mathcal{V} = V \otimes \mathbb{C}[[z^{-1}, z]$ be the space of V-valued Laurent series of the form

$$\cdots + q_2^*(-z)^{-3} + q_1^*(-z)^{-2} + q_0^*(-z)^{-1} + q_0 z^0 + q_1 z^1 + q_2 z^2 + \dots,$$

where $q_i, q_i^* \in V$ and $q_i = 0$ for i large enough. Let $\mathcal{V}_+ := V \otimes \mathbb{C}[z]$ and $\mathcal{V}_- := V \otimes z^{-1}\mathbb{C}[z^{-1}]$.

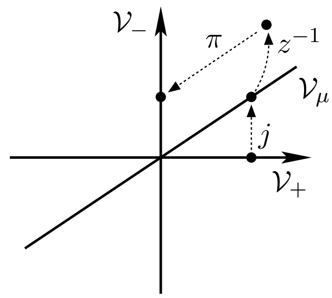
Any tower of endomorphisms $M_{a,b} : V \to V$, $a, b = 0, 1, \dots$, determines a linear map $\mu : \mathcal{V}_+ \to \mathcal{V}_-$:

$$\mu(q_0 + q_1 z + q_2 z^2 + \dots) = (-z)^{-1}\left(\sum_{i=0}^{\infty} M_{0,i} q_i\right) + (-z)^{-2}\left(\sum_{i=0}^{\infty} M_{1,i} q_i\right) + (-z)^{-3}\left(\sum_{i=0}^{\infty} M_{2,i} q_i\right) + \dots$$

Note that all the sums are actually finite.

Denote by $\mathcal{V}_\mu$ the graph of μ. It is a vector subspace of $\mathcal{V}$ transversal to $\mathcal{V}_-$. Let $j : \mathcal{V}_+ \to \mathcal{V}_\mu$ be the natural identification and let $\pi : \mathcal{V} \to \mathcal{V}_-$ be the projection to $\mathcal{V}_-$ along the graph of μ. Finally introduce the linear map $\varphi : \mathcal{V}_+ \to \mathcal{V}_-$ given by

$$\varphi = \pi \circ z^{-1} \circ j.$$



Now consider (a formal germ of) the trivial vector bundle $\mathcal{V} \times T$ over $(T, 0)$. The endomorphisms $M_{a,b}$ and the linear maps μ, j, π, and φ will all depend on $t \in T$.

Lemma 5.1. *The following two characterizations of a linear map $\mu : \mathcal{V}_+ \to \mathcal{V}_-$ are equivalent:*

(a) The matrices $M_{a,b}$ satisfy the master equations of Proposition 1.6;

(b) The image of φ is isomorphic to V and the differential $d_t\mu : \mathcal{V}_+ \otimes T \to \mathcal{V}_-$ factorizes through the map $\varphi \otimes \mathrm{id}$.

It is important to note that condition (b) depends solely on the graph of μ and its formulation involves only the vector space structure of $\mathcal{V}$ and the operator of multiplication by z^{-1}. Thus the loop group of $\mathrm{GL}(V)$, that is, the group of matrices $G(z) \in \mathrm{End}(V) \otimes (C)[[z^{-1}, z]$, which preserves both these structures, acts on the solutions of the commutativity equation.

Proof of Lemma 5.1. Let

$$Q = Q\left(\sum_{i=0}^{\infty} q_i z^i\right) = q_0 + \sum_{i \geq 0} M_{0,i} q_{i+1}.$$

This is a surjective linear map from $\mathcal{V}_+$ onto V.

Let us write out the maps φ and $\dot{\mu}$ in coordinates. We have

$$z^{-1}\, j\left(\sum_{i=0}^{\infty} q_i z^i\right) = \ldots + \left(\sum_{i=0}^{\infty} M_{1,i} q_i\right) z^{-3} - \left(\sum_{i=0}^{\infty} M_{0,i} q_i\right) z^{-2} + q_0 z^{-1} + q_1 + q_2 z + \ldots$$

The components of the decomposition of this vector along the graph of μ and along $\mathcal{V}_-$ are, respectively,

$$\ldots - \left(\sum_{i=0}^{\infty} M_{2,i} q_{i+1}\right) z^{-3} + \left(\sum_{i=0}^{\infty} M_{1,i} q_{i+1}\right) z^{-2} - \left(\sum_{i=0}^{\infty} M_{0,i} q_{i+1}\right) z^{-1} + q_1 + q_2 z + \ldots$$

and

$$\varphi\left(\sum_{i=0}^{\infty} q_i z^i\right) = \ldots + \left(M_{1,0} q_0 + \sum_{i=0}^{\infty} (M_{2,i} + M_{1,i+1}) q_{i+1}\right) z^{-3} - \left(M_{0,0} q_0 + \sum_{i=0}^{\infty} (M_{1,i} + M_{0,i+1}) q_{i+1}\right) z^{-2} + \left(q_0 + \sum_{i=0}^{\infty} M_{0,i} q_{i+1}\right) z^{-1}.$$

If the endomorphisms satisfy the master equations, then the latter expression is transformed into

$$\dots + M_{1,0}Qz^{-3} - M_{0,0}Qz^{-2} + Qz^{-1}.$$

Thus $\varphi\left(\sum_{i=0}^{\infty} q_i z^i\right)$ depends only on Q, *i.e.*, the image of φ is isomorphic to V. Conversely, since

$$\varphi(q_0) = \dots + M_{1,0}q_0z^{-3} - M_{0,0}q_0z^{-2} + q_0z^{-1},$$

if we want the image of φ to be isomorphic to V we must have

$$\varphi\left(\sum_{i=0}^{\infty} q_i z^i\right) = \dots + M_{1,0}Qz^{-3} - M_{0,0}Qz^{-2} + Qz^{-1}.$$

This implies the master equations

$$M_{a+1,b} + M_{a,b+1} = M_{a,0}M_{0,b}.$$

The map $\dot\mu$ is given by

$$\dot\mu\left(\sum_{i=0}^{\infty} q_i z^i\right) = \dots - \left(\sum_{i=0}^{\infty} \dot M_{2,i}q_i\right) z^{-3} + \left(\sum_{i=0}^{\infty} \dot M_{1,i}q_i\right) z^{-2} - \left(\sum_{i=0}^{\infty} \dot M_{0,i}q_i\right) z^{-1}.$$

If the endomorphisms $M_{a,b}$ satisfy the master equations, this is transformed into

$$\dots - \dot M_{2,0}Qz^{-3} + \dot M_{1,0}Qz^{-2} - \dot M_{0,0}Qz^{-1}.$$

Thus it depends only on Q and therefore factorizes through φ. Conversely, since

$$\dot\mu(q_0) = \dots - \dot M_{2,0}q_0z^{-3} + \dot M_{1,0}q_0z^{-2} - \dot M_{0,0}q_0z^{-1},$$

if we want the map $\dot\mu$ to factorize through φ we must have

$$\dot\mu\left(\sum_{i=0}^{\infty} q_i z^i\right) = \dots - \dot M_{2,0}Qz^{-3} + \dot M_{1,0}Qz^{-2} - \dot M_{0,0}Qz^{-1}.$$

This implies the master equations $\dot M_{a,b+1} = \dot M_{a,0}M_{0,b}$.

The last master equation $\dot M_{a+1,b} = M_{a,0}\dot M_{0,b}$ follows from the other two. □

5.2. **Symplectic framework.** The symplectic framework for the linear algebraic description of the master equations is important, because it allows one to obtain the formulas for the $\widehat{r}$-action (Equation (2) in Proposition 2.4) and the $\widehat{s}$-action (Equation (3) in Definition 3.1) as the result of the Weyl quantization of quadratic hamiltonians.

In order to put the description given above into a setup suitable for quantization, we have to double the loop space of V. Namely, consider $\mathbb{V} := (V \oplus V^*) \otimes \mathbb{C}[[z^{-1}, z]$. Let $\Omega(f,g) := \oint \langle f(-z), g(z)\rangle dz$, where $\langle\cdot,\cdot\rangle$ is the standard pairing of vectors and covectors in $V \oplus V^*$.

There is a natural action of the loop group of $\mathrm{GL}(V)$ on $\mathbb{V}$. This is the maximal group that preserves the operator of multiplication by z and the splitting of $\mathbb{V}$ into the direct sum of $V \otimes \mathbb{C}[[z^{-1}, z]$ and $V^* \otimes \mathbb{C}[[z^{-1}, z]$. The action is completely determined by its restriction to $V \otimes \mathbb{C}[[z^{-1}, z]$, where we have the same action as in the previous section.

$\mathbb{V}$ is naturally identified with $T^*\mathbb{V}_+$, where $\mathbb{V}_+ = (V \oplus V^*) \otimes \mathbb{C}[z]$. We view a full Gromov-Witten potential $F(p,q,t) = \sum M_{a,b}(t) p_a q_b$ as a function on $\mathbb{V}_+$ depending on an extra set of parameters $t \in T$. Introduce the maps

$$\begin{array}{cccc} \mu & : & \mathcal{V} \otimes C[z] & \to & \mathcal{V} \otimes z^{-1} C[[z^{-1}]] \\ & & \sum q_b z^b & \mapsto & \sum\limits_{a,b} (-z)^{-a-1} M_{a,b} q_b, \end{array}$$

and

$$\begin{array}{cccc} \mu^* & : & \mathcal{V}^* \otimes C[z] & \to & \mathcal{V}^* \otimes z^{-1} C[[z^{-1}]] \\ & & \sum p_a z^a & \mapsto & \sum\limits_{a,b} (-z)^{-b-1} p_a M_{a,b}. \end{array}$$

(The map μ is the same as in the previous section.) Then the graph of dF is a Lagrangian subspace of $\mathbb{V}$ that is equal to $\mathcal{V}_\mu \oplus \mathcal{V}^*{}_{\mu^*}$, where $\mathcal{V}_\mu$ and $\mathcal{V}^*{}_{\mu^*}$ are the graphs of μ and μ^*. Note that $\mathcal{V}^*{}_{\mu^*}$ is also unambiguously reconstructed from the condition that $\mathcal{V}_\mu \oplus \mathcal{V}^*{}_{\mu^*}$ is Lagrangian and, conversely, V_μ is the intersection of the graph of dF with $V \otimes \mathbb{C}[[z^{-1}, z]$.

Thus a power series $F(p,q,t) = \sum M_{a,b}(t) p_a q_b$ satisfies the master equations of Proposition 1.6 if and only if the intersection of the graph of dF with $V \otimes \mathbb{C}[[z^{-1}, z]$ satisfies condition (b) of Lemma 5.1. This condition is preserved by the loop group action.

Let us define the Weyl quantization of a quadratic function on $\mathbb{V}$. Let (e_μ) be a basis of V and (e^μ) the dual basis of V^*. An element of $\mathbb{V}$ can be written in coordinates as

$$\sum_{a\geq 0} p_{a,\mu} e^\mu z^a + \sum_{a \geq 0} \bar{p}_{a,\mu} e^\mu (-z)^{-a-1} + \sum_{b \geq 0} q_b^\mu e_\mu z^b + \sum_{a\geq 0} \bar{q}_b^\mu e_\mu (-z)^{-b-1}.$$

Thus we have $\Omega = \sum\limits_{a \geq 0} (d\bar{p}_{a,\mu} \wedge dq_a^\mu + d\bar{q}_a^\mu \wedge dp_{a,\mu})$. The Weyl quantization is then defined by the correspondence:

$$\bar{p}_{a,\mu} \mapsto \frac{\partial}{\partial q_a^\mu}; \qquad p_{a,\mu} \mapsto p_{a,\mu}; \qquad \bar{q}_b^\nu \mapsto \frac{\partial}{\partial p_{b,\nu}}; \qquad q_b^\nu \mapsto q_b^\nu;$$

together with the convention that the derivation operators are always placed to the right of the multiplication operators.

Now we can describe a way to obtain formulas for $\widehat{r}$-action (Equation (2) in Proposition 2.4) and $\widehat{s}$-action (Equation (3) in Definition 3.1) on the exponent of the full Gromov-Witten potential $\exp F(p,q,t)$. First, we consider the symplectic action of $\exp(s)$ and $\exp(r)$, $s = \sum_{l=1}^\infty s_l z^{-l}$ and $r = \sum_{l=1}^\infty r_l z^l$, $s_l, r_l \in \mathrm{End}(V)$, $i = 1, 2, \dots$. We obtain exponents of linear Hamiltonian vector fields. The corresponding Hamiltonians, H_s and H_r are quadratic and can be quantized according to the above conventions. The quantized Hamiltonians, $\widehat{H}_s$ and $\widehat{H}_r$ are differential operators of the first and second order respectively.

Theorem 5.2. *The action of $\widehat{H}_s$ and $\widehat{H}_r$ on the exponent of the full Gromov-Witten potential $F(p,q,t)$ is given by the formulas $\widehat{r}$-action (Equation (2) in Proposition 2.4) and $\widehat{s}$-action (Equation (3) in Definition 3.1).*

Proof. These formulas are obtained by a straightforward computation in the same way as it was done in [17]. $\square$

Remark 5.3. It is a general property of the Weyl quantization that the action of $\exp(s)$ and $\exp(r)$ on the graph of dF coincides with the action of $\exp(\widehat{H}_s)$ and $\exp(\widehat{H}_r)$ on $\exp F(p,q,t)$.

Remark 5.4. Unfortunately, there is no non-trivial higher genera theory for the commutativity equation. The only possible extension is to genus 1, where we can consider an extra function that depends only on t. But since there are no new relations coming from the geometry of the moduli space of genus 1 curves with only black marked points, the theory is empty there.

6. A link to the Losev-Polyubin action

In this section we have two goals. First, we recall a construction of the group action on solutions of commutativity equations due to Losev and Polyubin [24] (or rather we give our own interpretation of their construction with a new proof). Second, we prove a relation between the action that we develop in this paper and the Losev-Polyubin construction. This is a direct analog of the relation between the group actions constructed by van de Leur and Givental [10].

In this section we always assume that the number of t-variables coincides with the size of matrices ($\dim V = \dim T$). Unfortunately, we have to use certain standard definitions and basic theorems from the theory of multi-component KP hierarchies without prior explanation. We refer to [14, 15] for all preliminary material, in particular, we use the same notation as in these papers.

6.1. **Interpretation of the Losev-Polyubin action.** Losev and Polyubin associated in [24] a solution of the commutativity equation to an arbitrary invertible matrix formal power series $A(z) = A_0 + zA_1 + z^2A_2 + \dots$. Their construction has a nice interpretation in terms of wave functions of multi-component KP hierarchies. Moreover, while Losev and Polyubin give a formula only for $dM_{0,0}$, we can extend it in a natural way to the whole tower of descendant matrices $M_{a,b}$, $a, b \geq 0$.

Let $V^{\pm}(0,x,z)$ be the wave functions of multi-component KP hierarchies corresponding to the vector $A(z)|0\rangle$ (see the definition in [19, 10]). It is quite natural to consider the wave functions twisted by $A(z)$. We introduce the notation

$$\Psi^+(t,z) := V^+(0,x,z)A(z)|_{x_1=t, x_{\geq 2}=0};$$
$$\Psi^-(t,z) := A^{-1}(z)V^-(0,x,z)|_{x_1=t, x_{\geq 2}=0}.$$

We list the main properties of the matrices $\Psi^{\pm}(t,z)$:

- P1: $\Psi^{\pm}(t,z)$ is a matrix-valued formal power series in variables z and $t = (t^1, \dots, t^N)$.
- P2: $\Psi^-(t,-z)\Psi^+(t,z) = \mathrm{id}$.
- P3: The series $\Psi^+(t,z)$ satisfies the equation
$$\frac{\partial}{\partial t^k}\Psi^+(t,z) = (zE_{kk} + W_k)\Psi^+(t,z).$$
Here E_{kk} is the matrix with a 1 on the k-th diagonal entry and zeroes elsewhere, while $W_k = W_k(t)$ is some matrix that doesn't

depend on z. (In fact, W_k has a precise expression in terms of multi-component KP tau-functions corresponding to $A(z)$, but we don't need it.)

P4: $\Psi^-(t,z)$ satisfies the equation

$$\frac{\partial}{\partial t^k}\Psi^-(t,z) = -\Psi^-(t,z)(zE_{kk}+W_k).$$

We will now forget about the multi-component KP origin of the matrices $\Psi^\pm(t,z)$ and use the properties P1-P3 as axioms (P4 follows from P2 and P3).

One more piece of notation:

$$\Psi^+(t,z) = \Psi^+_0 + z\Psi^+_1 + z^2\Psi^+_2 + \dots;$$
$$\Psi^-(t,z) = \Psi^-_0 + z\Psi^-_1 + z^2\Psi^-_2 + \dots.$$

Theorem 6.1. (A generalization of Losev-Polyubin) *The matrices*

$$M_{a,b} := (-1)^b\Psi^-_{a+b+1}\Psi^+_0 + (-1)^{b-1}\Psi^-_{a+b}\Psi^+_1 + \dots + \Psi^-_{a+1}\Psi^+_b$$

satisfy the master equations of Proposition 1.6, that is, $M_{a+1,b}+M_{a,b+1} = M_{a,0}M_{0,b}$*,* $dM_{a+1,b} = M_{a,0}dM_{0,b}$*, and* $dM_{a,b+1} = dM_{a,0}M_{0,b}$*.*

In particular, $M_{0,0} = \Psi^-_0\Psi^+_1 = \Psi^-_1\Psi^+_0$, $M_{a,0} = \Psi^-_{a+1}\Psi^+_0$, $M_{0,b} = \Psi^-_0\Psi^+_{b+1}$. The original statement of Losev and Polyubin is equivalent to the following explicit formula for $dM_{0,0}$.

Proposition 6.2. *We have* $dM_{0,0} = \Psi^-_0 \operatorname{diag}(dt^1,\dots,dt^n)\Psi^+_0$*.*

Proof of Theorem 6.1 and Proposition 6.2. In order to prove the theorem it is enough to show that $M_{a+1,b}+M_{a,b+1} = M_{a,0}M_{0,b}$ and $dM_{0,b+1} = dM_{0,0}M_{0,b}$.

First, observe that P2 implies that $M_{0,b} = \Psi^-_0\Psi^+_{b+1}$. This means that $M_{a,0}M_{0,b}$ is equal to $\Psi^-_{a+1}\Psi^+_0\Psi^-_0\Psi^+_{b+1} = \Psi^-_{a+1}\Psi^+_{b+1}$ (we apply P2 again). On the other hand, in the expression for the sum $M_{a+1,b}+M_{a,b+1}$ all terms are cancelled except for $\Psi^-_{a+1}\Psi^+_{b+1}$.

P3 and P4 then imply that $\frac{\partial}{\partial t_k}M_{0,b}$ is equal to

$$\frac{\partial}{\partial t_k}\left(\Psi^-_0\Psi^+_{b+1}\right) = -\Psi^-_0W_k\Psi^+_{b+1} + \Psi^-_0E_{kk}\Psi^+_b + \Psi^-_0W_k\Psi^+_{b+1} = \Psi^-_0E_{kk}\Psi^+_b.$$

The proposition is proved by substituting $b=0$ in the last equality.

Finally, we apply P2 once again and we obtain

$$\frac{\partial}{\partial t_k}M_{0,b+1} = \Psi^-_0E_{kk}\Psi^+_{b+1} = \Psi^-_0E_{kk}\Psi^+_0\Psi^-_0\Psi^+_{b+1} = \frac{\partial(\Psi^-_0\Psi^+_1)}{\partial t_k}\Psi^-_0\Psi^+_{b+1},$$

which is equal to $\frac{\partial M_{0,0}}{\partial t_k}M_{0,b}$. □

Remark 6.3. Using the approach from the commutativity equation side it is easier to explain the result of van de Leur [19] than it is done in the original paper. He constructs a solution of the WDVV equation starting from $A(z)|0\rangle$ with $A(-z)^tA(z) = \mathrm{id}$ and passing through the Darboux-Egoroff system of equations in canonical coordinates.

Instead one can observe that if $A(-z)^tA(z) = id$, then $\Psi^-(z) = \Psi^+(-z)^t$, and the matrix $M_{0,0}$ happens to be symmetric. One can classify all possible

changes of variables such that $M_{0,0}$ turns into the matrix of second derivatives of some function [7]. This function is then a solution of the WDVV equation. One of the simplest changes of variables is $(t^1_{new},\dots,t^N_{new}) = (1,\dots,1)M_{0,0}$, and it is exactly the change of variables that van de Leur is applying in [19].

6.2. **Lie algebra action.** In the extension of the Losev-Polyubin construction discussed above we have $\Psi_i^\pm = \Psi_i^\pm(A(z))$ and $M_{a,b} = M_{a,b}(A(z))$. That is, the system of matrices depends on the choice of an invertible matrix-valued formal power series $A(z) = A_0 + A_1 z + A_2 z^2 + \dots$. Let $r(z) = r_1 z + r_2 z^2 + \dots$ be an arbitrary formal power series of matrices. We introduce the notation for the derivatives

$$r(z).M_{a,b}(A(z)) := \frac{\partial}{\partial\epsilon} M_{a,b}\left(A(z)\exp(\epsilon r(z))\right)\Big|_{\epsilon=0}, \qquad a,b\geq 0;$$
$$r(z).\Psi_i^\pm(A(z)) := \frac{\partial}{\partial\epsilon} \Psi_i^\pm\left(A(z)\exp(\epsilon r(z))\right)\Big|_{\epsilon=0}, \qquad i\geq 0.$$

The formula for $r(z).\Psi_k^+$ is computed in [10]:

$$(r_\ell z^\ell).\Psi_k^+ = \Psi^+_{\ell+k} r_l - \sum_{i=1}^{\ell}\sum_{j=0}^{\ell-i} (-1)^{\ell-i-j}\Psi_j^+ r_\ell \Psi^-_{\ell-i-j}\Psi^+_{i+k}.$$

It allows to compute explicitly all expressions for $r(z).M_{a,b}$.

Theorem 6.4. *The formulas for the Lie algebra action $r(z).M_{a,b}$ in Losev-Polyubin framework coincide with* (1) *up to a change of sign.*

Proof. Since all matrices $M_{a,b}$ are polynomial expressions in $M_{0,b}$, it is enough to prove the theorem for $M_{0,b} = \Psi_0^-\Psi^+_{b+1}$. Using P2, we have:

$$(r_\ell z^\ell).\left(\Psi_0^-\Psi^+_{b+1}\right) = -\Psi_0^-\left((r_\ell z^\ell).\left(\Psi_0^+\right)\right)\Psi_0^-\Psi^+_{b+1} + \Psi_0^-(r_\ell z^\ell).\left(\Psi^+_{b+1}\right).$$

Using P2 again we can rewrite the formula for $(r_\ell z^\ell).\left(\Psi_0^+\right)$ as

$$(r_\ell z^\ell).\left(\Psi_0^+\right) = \Psi_\ell^+ r_\ell + \sum_{i=0}^{\ell-1}(-1)^{\ell-q}\Psi_i^+ r_\ell\Psi^-_{\ell-i}\Psi_0^+.$$

Therefore,

$$\begin{aligned}
&(r_\ell z^\ell).\left(\Psi_0^-\Psi^+_{b+1}\right)\\
&\quad= -\Psi_0^-\Psi_\ell^+ r_\ell\Psi_0^-\Psi^+_{b+1} - \textstyle\sum_{j=0}^{\ell-1}(-1)^{\ell-j}\Psi_0^-\Psi_j^+ r_\ell\Psi^-_{\ell-j}\Psi^+_{b+1}\\
&\qquad +\Psi_0^-\Psi^+_{\ell+b+1} r_l - \textstyle\sum_{i=1}^{\ell}\sum_{j=0}^{\ell-i}(-1)^{\ell-i-j}\Psi_0^-\Psi_j^+ r_\ell\Psi^-_{\ell-i-j}\Psi^+_{i+b+1}\\
&\quad= M_{0,b+\ell}r_\ell - \textstyle\sum_{i=0}^{\ell}\sum_{j=0}^{\ell-i}(-1)^{\ell-i-j}\Psi_0^-\Psi_j^+ r_\ell\Psi^-_{\ell-i-j}\Psi^+_{i+b+1}\\
&\quad= M_{0,b+\ell}r_\ell + \textstyle\sum_{j=1}^{\ell}(-1)^{\ell-j-1}M_{0,j-1}r_\ell M_{\ell-j,b} + (-1)^{\ell-1}r_\ell M_{\ell,b}.
\end{aligned}$$

The last expression coincides with (1) up to multiplication by $(-1)^{\ell-1}$. $\square$

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